GEOMETRIC REPRESENTATION THEORY OF THE HILBERT SCHEMES PART II

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Abstract. Identifying the sum of (equivariant) homology groups of $(\mathbb{C}^2)^{[n]}$ with the Fock space, we interpret geometrically some important elements of the Fock space. As a corollary, we prove an existence of Jack polynomials.

1. Recollection

In today's lecture we use the following notation:

$\circ$ $X = \mathbb{C}^2$.
$\circ$ $s : X^{[n]} \to \mathrm{Sym}^n X$ is the Hilbert-Chow map.
$\circ$ $T = \mathbb{C}^* \times \mathbb{C}^*$ is the two-dimensional torus acting on X and, therefore, on $X^{[n]}$ and $\mathrm{Sym}^n X$.
$\circ$ $\xi_\lambda \in X^{[n]}$ denotes the T-fixed point parametrized by the Young diagram λ.
$\circ$ λ^* denotes the conjugate of the Young diagram λ.
$\circ$ $\mathcal{H}$ denotes the Heisenberg algebra.
$\circ$ $M := \bigoplus H_*(X^{[n]}),\ M^T := \bigoplus H_*^{T,BM}(X^{[n]}),\ M^T_{\mathrm{loc}} := \bigoplus H_*^{T,BM}(X^{[n]})_{\mathrm{loc}}$.
$\circ$ $R := H_T^*(\mathrm{pt}) = \mathbb{C}[\epsilon_1, \epsilon_2],\ \mathbb{F} := \mathrm{Frac}(R) = \mathbb{C}(\epsilon_1, \epsilon_2)$, where ϵ_1, ϵ_2 form a natural basis of $\mathrm{Lie}\,T$, corresponding to the one-dimensional subtori $\{(t, 1)\}$ and $\{(1, t)\}$, respectively.

Last time we constructed an action of $\mathcal{H}$ on M by using the Grojnowski-Nakajima correspondences $Z_\alpha[i]$ and $Z_\beta[j]$. We also proved that M is isomorphic to a Fock module over $\mathcal{H}$. In other words, there exists an isomorphism of $\mathcal{H}$-modules

$$\theta : \mathbb{C}[z_1, z_2, \ldots] \xrightarrow{\sim} M,$$

where $\mathbb{C}[z_1, z_2, \ldots]$ is a level 1 Fock module over $\mathcal{H}$, and $\theta(1) = \mathbf{1}$–the generator of $H_0(X^{[0]})$. This isomorphism depends on the nonzero class $\beta \in H_0(X) \simeq \mathbb{C}[\mathrm{pt}]$, namely:

$$\theta(z_{i_1} z_{i_2} \cdots z_{i_N}) = Z_\beta[-i_1] Z_\beta[-i_2] \cdots Z_\beta[-i_N](\mathbf{1}) \ \ \forall\ i_1 \geq i_2 \geq \cdots \geq i_N.$$

We also proved that the same correspondences define an action of $\mathcal{H}$ on M^T and M^T_{loc}. According to the localization theorem:

$$M^T_{\mathrm{loc}} \simeq \bigoplus_\lambda \mathbb{F} \cdot [\xi_\lambda].$$

Since $\mathbf{1} \in H_0^{T,BM}(X^{[0]})$ is annihilated by $\{Z_\alpha[i]\}_{i>0}$ and M^T_{loc} has the same q-dimension as the Fock module, we actually get an isomorphism of $\mathcal{H}$-modules

$$\theta^T : \mathbb{F}[z_1, z_2, \ldots] \xrightarrow{\sim} M^T_{\mathrm{loc}},$$

defined in the same way as θ for any nonzero class $\beta \in H_*^{T,BM}(X)$.

Remark 1.1. (a) The Poincaré dual of $[x-\text{axis}]$ and $[y-\text{axis}]$ are actually $\epsilon_2 \cdot 1$ and $\epsilon_1 \cdot 1$.
(b) Note that $H_*^T(X) \simeq H_T^*(\mathrm{pt}) \cdot [0],\ H_*^{T,BM}(X) \simeq H_T^*(\mathrm{pt}) \cdot [X]$, since $\mathbb{C}^2 \times_T ET \to BT$ is a vector bundle. Also $H_*^T(X)_{\mathrm{loc}} \simeq \mathbb{F} \cdot [0],\ H_*^{T,BM}(X)_{\mathrm{loc}} \simeq \mathbb{F} \cdot [X]$ by the localization theorem. Therefore, the choice of α, β is unique up to proportionality.

2. Symmetric functions

2.1. **Ring Λ.**

Fix $N \in \mathbb{N}$ and let Λ_N be the ring of symmetric functions in N variables $x_1, \ldots, x_N$, that is,

$$\Lambda_N := \mathbb{Z}[x_1, \ldots, x_N]^{S_N}.$$

This ring is naturally graded by the degree of polynomials:

$$\Lambda_N = \bigoplus_{n \geq 0} \Lambda_N^n.$$

For any $K > N$, there is a homomorphism

$$\mathbb{Z}[x_1, \ldots, x_K] \to \mathbb{Z}[x_1, \ldots, x_N] \text{ given by } x_1 \mapsto x_1, \ldots, x_N \mapsto x_N, x_{N+1} \mapsto 0, \ldots, x_K \mapsto 0.$$

It induces the homomorphism of graded rings

$$\rho_{K,N} : \Lambda_K \to \Lambda_N.$$

Let us point out that for any $K > N \geq n$, the degree n component of $\rho_{K,N}$ is actually an isomorphism

$$\rho_{K,N}^n : \Lambda_K^n \xrightarrow{\sim} \Lambda_N^n.$$

Therefore, we can define the ring of symmetric functions in infinitely many variables as

$$\Lambda := \bigoplus_{n \geq 0} \Lambda^n \text{ with } \Lambda^n := \varprojlim \Lambda_N^n.$$

Finally, we define $\Lambda_R := \Lambda \otimes_{\mathbb{Z}} R$ for any ring R.

2.2. **Two bases for $\Lambda_{\mathbb{Q}}$.**

Recall the two families of symmetric functions:

- *Monomial symmetric functions m_λ.*

Fix a Young diagram λ. For $N \geq l(\lambda) = \lambda_1^*$, define $m_\lambda \in \Lambda_N^{|\lambda|}$ by

$$m_\lambda(x_1, \ldots, x_N) := \frac{1}{\#\{\sigma \in S_N : \sigma(\lambda) = \lambda\}} \sum_{\sigma \in S_N} x_1^{\lambda_{\sigma(1)}} \cdots x_N^{\lambda_{\sigma(N)}}.$$

For any $K > N \geq l(\lambda)$, we have

$$\rho_{K,N}^{|\lambda|}(m_\lambda(x_1, \ldots, x_K)) = m_\lambda(x_1, \ldots, x_N).$$

Thus, the sequence $\{m_\lambda(x_1, \ldots, x_N)\}_{N \geq l(\lambda)}$ defines an element of Λ, which we denote by m_λ.

It is well known that $\{m_\lambda\}_\lambda$ is a basis for Λ, and hence also for $\Lambda_{\mathbb{Q}}$.

- *Power symmetric functions p_λ.*

Let us consider the n-th power sums

$$p_n := m_{(n)} = \sum x_i^n \in \Lambda.$$

We define $p_\lambda \in \Lambda$ by

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots .$$

It is well known that $\{p_\lambda\}_\lambda$ is a basis for $\Lambda_{\mathbb{Q}}$ (but not for Λ).

Identifying $\Lambda_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}[p_1, p_2, \ldots]$, we will view the isomorphism θ^T as

$$(\star) \qquad \theta^T : \Lambda_{\mathbb{F}} \xrightarrow{\sim} M_{\mathrm{loc}}^T.$$

3. Geometric realization of m_λ

In this section we describe geometrically the images of $m_\lambda \in \Lambda_{\mathbb{F}}$ under the isomorphism $(\star)$.

3.1. Subvarieties $L^\lambda\Sigma$.

Let $\Sigma \subset X$ denote the x-axis, i.e., $\Sigma = \{(*, 0)\} \subset \mathbb{C}^2$.

Definition 3.1. Define $L^*\Sigma \subset \bigsqcup_n X^{[n]}$ as the locus, corresponding to those ideals $I \subset \mathbb{C}[x,y]$ such that $\operatorname{supp}(\mathbb{C}[x,y]/I) \subset \Sigma$.

In other words, $L^*\Sigma = \bigsqcup_n s^{-1}(\operatorname{Sym}^n \Sigma)$. Note that $\operatorname{Sym}^n \Sigma$ has a natural stratification

$$\operatorname{Sym}^n \Sigma = \bigsqcup_{\lambda \vdash n} S^n_\lambda \Sigma, \; S^n_\lambda\Sigma := \left\{ \sum \lambda_i [x_i] \in \operatorname{Sym}^n \Sigma \mid x_i \neq x_j \text{ for } i \neq j \right\}.$$

Exercise 3.1. *Show that $s^{-1}(S^n_\lambda \Sigma)$ are locally closed n-dimensional irreducible subvarieties of $L^n\Sigma := L^*\Sigma \cap X^{[n]}$.*

Moreover, their closures

$$L^\lambda\Sigma := \overline{s^{-1}(S^n_\lambda\Sigma)}$$

are irreducible components of $L^*\Sigma$. Next, we provide alternative definitions of $L^\lambda\Sigma$.

3.2. $L^\lambda\Sigma$ via a $\mathbb{C}^*$-action.

Let us consider a one dimensional subtorus $T' \subset T$ given by $T' = \{(1,t)\}$. Then we have:

Proposition 3.2. *For a point $\xi \in X^{[n]}$, there exists a limit $\lim\limits_{t\to\infty} (1,t) \cdot \xi$ iff $\xi \in L^n\Sigma$.*

Proof. Follows from the properness of s and an analogous result for $\operatorname{Sym}^n X$. □

For a Young diagram λ and $z_0 \in \Sigma$, let $I_{\lambda, z_0} \subset \mathbb{C}[x,y]$ be the ideal parametrized by λ and such that $\operatorname{supp}(\mathbb{C}[x,y]/I_{\lambda,z_0}) = \{(z_0, 0)\}$, that is,

$$I_{\lambda,z_0} := (y^{\lambda_1}, (x - z_0)y^{\lambda_2}, \ldots, (x - z_0)^{\lambda^*_1}).$$

The following is obvious:

Proposition 3.3. [N1, Proposition 7.4] *If a codimension n ideal $I \subset \mathbb{C}[x,y]$ defines a T'-fixed point of $X^{[n]}$, then it can be uniquely expressed as $I = I_{\lambda^1,z_1} \cap \cdots \cap I_{\lambda^r,z_r}$ for r distinct points $z_1, \ldots, z_r \in \Sigma$ and a collection of Young diagrams $\{\lambda^i\}$ such that $\sum |\lambda^i| = n$. Conversely, any such intersection $I_{\lambda^1,z_1} \cap \cdots \cap I_{\lambda^r,z_r}$ defines a T'-fixed point of $X^{[n]}$.*

For a collection $\{\lambda^1, \ldots, \lambda^r\}$ of r Young diagrams we associate a single Young diagram λ, defined by $\lambda = \lambda^1 \cup \ldots \cup \lambda^r$. In other words, if $\lambda^j = (1^{n^j_1} 2^{n^j_2} \ldots)$, then $\lambda = (1^{n^1_1 + \ldots + n^r_1} 2^{n^1_2 + \ldots + n^r_2} \ldots)$.

Exercise 3.4. *Verify that $I_{\lambda^1,z_1} \cap I_{\lambda^2,z_2} \to I_{\lambda^1\cup\lambda^2,z_1}$ as $z_2 \to z_1$.*

For a Young diagram $\lambda = (1^{n_1} 2^{n_2} \ldots)$, we define $S^\lambda\Sigma$ as the locus of $(X^{[n]})^{T'}$ such that the associated collection $\{\lambda^1, \ldots, \lambda^r\}$ satisfies $\lambda = \lambda^1 \cup \ldots \cup \lambda^r$. Together with Exercise 3.4, we get:

Proposition 3.5. *(a) $S^\lambda\Sigma = S^{n_1}\Sigma \times S^{n_2}\Sigma \times \ldots$.*
(b) The irreducible components of $(X^{[n]})^{T'}$ are exactly $\{S^\lambda\Sigma\}_{\lambda \vdash n}$.
(c) Each $S^\lambda\Sigma$ has an open stratum $S^\lambda_0\Sigma$ corresponding to $\lambda^1, \ldots, \lambda^r$ being 1-column diagrams.

Consider the decomposition $L^n\Sigma = \bigsqcup_{\lambda\vdash n} W^-_\lambda$, $W^-_\lambda := \{\xi \in L^n\Sigma \mid \lim\limits_{t\to\infty} (1,t) \cdot \xi \in S^\lambda\Sigma\}$.

Proposition 3.6. [N3, Proposition 2.17] *We have $L^\lambda\Sigma = \overline{W^-_\lambda}$.*

Proof. Follows from $S^\lambda_0\Sigma \subset s^{-1}(S^n_\lambda\Sigma)$ (both $L^\lambda\Sigma, \overline{W^-_\lambda}$ are irreducible and equidimensional). □

Proposition 3.7. *For any diagram λ, the component $L^\lambda\Sigma$ is a Lagrangian subvariety of $X^{[n]}$.*

Proof. Note that the symplectic form ω on $X^{[n]}$ is semi-invariant w.r.t. T'-action: $\psi_t^*\omega = t\cdot\omega$. For any $\xi\in S^\lambda\Sigma$, consider a weight decomposition of the tangent space: $T_\xi X^{[n]} = \oplus_n H_n$. The above condition implies $H_n \perp^\omega H_m$ unless $n+m=1$. Together with the nondegeneracy of ω, we see that $T_{\xi_\lambda}W_\lambda^- = \oplus_{n\le 0}H_n$ has half dimension. Further, for any $y\in W_\lambda^-$ close to x and $u,v\in T_yW_\lambda^-$, we get $\omega_{ty}(tu,tv) = t\cdot\omega_y(u,v)$. Existence of $\lim\limits_{t\to\infty} t\omega_y(u,v)$ implies $\omega(u,v)=0$. □

For any $m,l\in\mathbb{N}$, consider a one-dimensional subtorus $T_{m,l} := \{(t^{-m},t^l)\}$ of T. For a fixed n and *generic* $m,l\in\mathbb{N}$ we have $(X^{[n]})^{T_{m,l}} = (X^{[n]})^T$.[1]

Proposition 3.8. *(a) For a point $\xi\in X^{[n]}$, there exists a limit $\lim\limits_{t\to\infty}(t^{-m},t^l)\cdot\xi$ iff $\xi\in L^n\Sigma$.*
(b) We also have $W_\lambda^- := \{\xi\in L^n\Sigma \mid \lim\limits_{t\to\infty}(t^{-m},t^l)\cdot\xi = \xi_\lambda\}$.

The proof of part (b) relies on the character formula from the end of last talk:

$$(\dagger)\qquad \mathrm{ch}\, T_{\xi_\lambda}(X^{[n]}) = \sum_{\square\in\lambda}\left(t_1^{l(\square)+1}t_2^{-a(\square)} + t_1^{-l(\square)}t_2^{a(\square)+1}\right).$$

Proof. (a) Same as in Proposition 3.2.
(b) Both varieties are T-invariant, so it suffices to check the equality in the neighborhood of ξ_λ. In such a neighborhood, the contractable locus corresponds to the sum of non-positive weight spaces. However, a T-weight from ($\dagger$) is either both T' and $T_{m,l}$ positive or non-positive. □

The benefit of $T_{m,l}$-action rather then T'-action is that the fixed point locus is discrete.[2]

3.3. **Geometric realization of m_λ.**
Let N^T be the sum of the Borel-Moore equivariant homology groups of $L^*\Sigma$:

$$N^T := H_*^{T,BM}(L^*\Sigma) = \bigoplus H_*^{T,BM}(L^n\Sigma) = \bigoplus \mathbb{F}\cdot[L^\lambda\Sigma].$$

If $\alpha=\epsilon_1,\ \beta=\epsilon_2$ are the Poincaré dual to $[y-\text{axis}]$ and $[x-\text{axis}]$, then the correspondences $Z_\alpha[i]$ and $Z_\beta[-i]$ also act on N^T.[3] Analogously to ($\star$), we have an isomorphism $\vartheta^T:\Lambda_\mathbb{F}\xrightarrow{\sim}N^T_{\mathrm{loc}}$

$$p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots\mapsto Z_\beta[-\lambda_1]Z_\beta[-\lambda_2]\cdots\mathbf{1}\ \ \forall\ \lambda_1\ge\lambda_2\ge\cdots.$$

Proposition 3.9. *We have $\vartheta^T: m_\lambda\mapsto[L^\lambda\Sigma]$.*

Sketch of the proof. This result is a generalization of the corresponding fact in a non-equivariant setting [N1, Theorem 9.14]. However, the latter should be applied to the compactification $\mathbb{P}^2$, rather then $\mathbb{C}^2$ itself, since Σ defines a zero homology class of $\mathbb{C}^2$.

To check $\vartheta^T(m_\lambda) = [L^\lambda\Sigma]$, it suffices to prove $Z_\Sigma[-i][L^\lambda\Sigma] = \sum_\mu a_{\lambda\mu}[L^\mu\Sigma]$, where the coefficients $a_{\lambda\mu}$ are determined by the identity $p_i\cdot m_\lambda = \sum_\mu a_{\lambda\mu}m_\mu$ in Λ. It is clear that $a_{\lambda\mu}$ is equal to the number of indexes r such that $\{\lambda_1,\ldots,\lambda_{r-1},\lambda_r+i,\lambda_{r+1},\ldots\}=\{\mu_1,\mu_2,\ldots\}$.

In order, to determine the coefficient of $[L^\mu\Sigma]$ in $Z_\Sigma[-i][L^\lambda\Sigma]$, we can compute everything in the neighborhood of an arbitrary point $J_0\in L^\mu\Sigma$. We choose such a point to be of the form $J_0 = I_{\mu_1,z_1}\cap\cdots\cap I_{\mu_l,z_l}$ for pairwise distinct points $z_1,\ldots,z_l\in\Sigma$, $l:=l(\mu)$.

Then $(J_0,J,x)\in Z[-i]\iff\exists\, j: x=x_j$ and $J = I_{\mu_1,z_1}\cap\cdots\cap I_{\mu_j-i,z_j}\cap\cdots\cap I_{\mu_l,z_l}$. Therefore, the coefficient of $[L^\mu\Sigma]$ in $Z_\Sigma[-i][L^\lambda\Sigma]$ is nonzero iff $a_{\lambda\mu}\ne 0$. In the latter case $a_{\lambda\mu}$ is equal to the number of possible choices of $x\in X$ as above. It remains only to check that each such choice of x contributes 1 to the coefficient. This requires a transversality result (see [N1, p.112]). □

[1] A similar argument was already used last time in the proof of $\dim_q M = \prod_{j=1}^\infty \frac{1}{1-q^j}$.
[2] In [N3], Nakajima considers only $T_{1,1}$. However, it is not obvious for us why $(X^{[n]})^{T_{1,1}} = (X^{[n]})^T$.
[3] Those classes are nonzero in the equivariant homology, unlike in the non-equivariant setting.

4. Geometric realization of Jack polynomials

In this section we introduce the important class of symmetric functions called Jack polynomials. Using the isomorphism ($\star$), we provide their geometric interpretation. In particular, this yields an alternative proof of their existence. Our exposition follows [LQW, N3].

4.1. Jack polynomials $P_\lambda^{(k)}$.

Let k be an independent variable. Consider the inner product $\langle \cdot, \cdot \rangle_k$ on $\Lambda_{\mathbb{Q}(k)}$ defined by

$$\langle p_\lambda, p_\mu \rangle_k := k^{l(\lambda)} z_\lambda \delta_\lambda^\mu,$$

where $z_\lambda := \prod l^{n_l} n_l!$ for $\lambda = (1^{n_1} 2^{n_2} \cdots)$.

Last time we introduced a complete order $\preceq$ and a partial order $\leq$ on Young diagrams.

Theorem 4.1. *For each partition λ, there is a unique symmetric polynomial $P_\lambda^{(k)}$ satisfying:*
(i) $P_\lambda^{(k)} = m_\lambda + \sum_{\mu<\lambda} u_{\lambda,\mu}^{(k)} m_\mu$ for some $u_{\lambda,\mu}^{(k)} \in \mathbb{Q}(k)$.
(ii) $\langle P_\lambda^{(k)}, P_\mu^{(k)} \rangle_k = 0$ if $\lambda \neq \mu$.

Definition 4.1. Polynomials $P_\lambda^{(k)}$ are called the *Jack polynomials*.

Remark 4.1. For $k = 1$ we recover back the Schur polynomials: $P_\lambda^{(1)} = s_\lambda$.

The uniqueness of the orthogonal basis $\{P_\lambda^{(k)}\}_\lambda$ is clear from the Gram-Schmidt orthogonalization process. Namely, there exists a unique basis $\{P_\lambda^{(k)}\}$ satisfying condition (ii) and
(i') $P_\lambda^{(k)} = m_\lambda + \sum_{\mu \prec \lambda} u_{\lambda,\mu}^{(k)} m_\mu$ for some $u_{\lambda,\mu}^{(k)} \in \mathbb{Q}(k)$.

However, it is quite nontrivial to show that $u_{\lambda,\mu}^{(k)} = 0$ unless $\mu < \lambda$ (see [M, Section VI.10]).

Remark 4.2. The original proof is based on the following idea. One can construct a family of pairwise commuting differential operators $\{D_i\}$ acting on Λ, which are self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_k$. It is easy to check that $D_i(m_\lambda)$ is a linear combination of $\{m_\mu\}_{\mu \leq \lambda}$ and $\{D_i\}$ have a simple spectrum. Therefore, their joint eigenvectors (properly normalized) satisfy (i) and (ii).

We also introduce the *integral form* $J_\lambda^{(k)}$ of the Jack polynomials by

$$J_\lambda^{(k)} := c_\lambda(k) P_\lambda^{(k)}, \text{ where } c_\lambda(k) := \prod_{\Box \in \lambda} (k \cdot a(\Box) + l(\Box) + 1).$$

Remark 4.3. It turns out that $J_\lambda^{(k)}$ is a linear combination of $\{m_\mu\}_{\mu \leq \lambda}$ with coefficients in $\mathbb{Z}_{\geq 0}[k]$. Therefore, one can specialize k to any complex number in $J_\lambda^{(k)}$, but not in $P_\lambda^{(k)}$.

4.2. Geometric realization of $P_\lambda^{(k)}$.

In this section we provide a geometric realization of the Jack polynomials. It is worth to mention that this construction has no counterpart in the non-equivariant setting, unlike p_λ, m_λ.

Let us start from the following sequence of isomorphisms:

$$\bigoplus_{\lambda \vdash n} \mathbb{F} \cdot [\xi_\lambda] = H_*^T((X^{[n]})^T)_{\mathrm{loc}} \xrightarrow[\iota_*]{\sim} H_*^{T,BM}(L^n\Sigma)_{\mathrm{loc}} \xrightarrow[\jmath_*]{\sim} H_*^{T,BM}(X^{[n]})_{\mathrm{loc}},$$

where $\jmath : L^n\Sigma \hookrightarrow X^{[n]}$, $\iota : \bigsqcup_{\lambda \vdash n} \{\xi_\lambda\} \hookrightarrow L^n\Sigma$, $\iota_\lambda : \{\xi_\lambda\} \hookrightarrow X^{[n]}$ are the inclusions.

Note that $\{[L^\lambda\Sigma]\}_{\lambda \vdash n}$ is a natural basis of $H_*^{T,BM}(L^n\Sigma)_{\mathrm{loc}}$. Our next goal is to compute $\iota_*^{-1}([L^\lambda\Sigma])$ in the fixed point basis $\{[\xi_\mu]\}$. By the fixed point formula, we have

$$\iota_*^{-1}([L^\lambda\Sigma]) = \sum_{\mu : \xi_\mu \in L^\lambda\Sigma} c_{\lambda,\mu}[\xi_\mu], \; c_{\lambda,\mu} \in \mathbb{F}. \tag{1}$$

Remark 4.4. If ξ_μ is a smooth point of $L^\lambda\Sigma$, then $c_{\lambda,\mu}=\frac{1}{e(T_{\xi_\mu}L^\lambda\Sigma)}$, where $e(T_{\xi_\lambda}L^\lambda\Sigma)$ denotes the Euler class of the corresponding tangent space.

The following result provides a geometric interpretation of the dominance order on Young diagrams. We postpone its proof until the end of this section.

Proposition 4.2. *If $\xi_\mu\in L^\lambda\Sigma$, then $\mu\le\lambda$. Moreover, ξ_λ is a smooth point of $L^\lambda\Sigma$.*

Let us now consider the intersection pairing

$$\langle\cdot,\cdot\rangle : H_*^{T,BM}(X^{[n]})\otimes H_*^T(X^{[n]})\to H_*^T(\mathrm{pt}),\ u\otimes v\mapsto (-1)^n p_{X^{[n]}*}(u\cap v).$$

This pairing is perfect, due to the Poincaré duality, and yields a perfect pairing[4]

$$\langle\cdot,\cdot\rangle : M^T_{\mathrm{loc}}\otimes M^T_{\mathrm{loc}}\to\mathbb{F}.$$

Moreover, we have:[5]

$$\langle Z_\alpha[i]u,v\rangle=\langle u,Z_\alpha[-i]v\rangle,\ Z_{f\alpha}[i]=fZ_\alpha[i],\ Z_\alpha[i]f=fZ_\alpha[i],\ f\in H_T^*(\mathrm{pt}).$$

The first equality implies

$$\langle P_\lambda[\alpha],P_\mu[\beta]\rangle=(-\langle\alpha,\beta\rangle)^{l(\lambda)}z_\lambda\delta_\lambda^\mu,\ \text{where } P_\mu[\beta]:=Z_\beta[-\mu_1]Z_\beta[-\mu_2]\ldots(\mathbf{1}). \tag{2}$$

In other words, the isomorphism θ^T intertwines $\langle,\rangle_k$ on the $\Lambda_{\mathbb{F}}$-side with $\langle,\rangle$ on the M^T_{loc}-side, where $k=-\langle\beta,\beta\rangle$. In particular, for $\beta=\epsilon_2$ we get $k=-\epsilon_2/\epsilon_1$.[6]

Note that the intersection pairing $\langle,\rangle_T$ on $H_*^T((X^{[n]})^T)_{\mathrm{loc}}=\bigoplus_{\lambda\vdash n}\mathbb{F}\cdot[\xi_\lambda]$ is a direct sum of those on $H_*^T(\{\xi_\lambda\})_{\mathrm{loc}}$, that is, $\langle[\xi_\lambda],[\xi_\mu]\rangle_T=\delta_\lambda^\mu$. On the other hand, by the projection formula:

$$\langle \jmath_*\iota_*(A),\jmath_*\iota_*(B)\rangle=\langle A,\iota^*\jmath^*\jmath_*\iota_*B\rangle.$$

Since $\iota_\lambda^*\iota_{\lambda*}(\bullet)=e(T_{\xi_\lambda}X^{[n]})\cap\bullet$, we get $\langle[\xi_\lambda],[\xi_\mu]\rangle=(-1)^n e(T_{\xi_\lambda}X^{[n]})\cdot\delta_\lambda^\mu$.

Combining this observation with Proposition 4.2 and formulas (1)-(2), we get

Theorem 4.3. *Under the isomorphism $\theta^T:\Lambda_{\mathbb{F}}\xrightarrow{\sim}M^T_{\mathrm{loc}}$, we have*

$$P_\lambda^{(k)}\mapsto\frac{1}{e(T_{\xi_\lambda}L^\lambda\Sigma)}[\xi_\lambda],\ k=-\epsilon_2/\epsilon_1.$$

Remark 4.5. This theorem also proves an existence of the Jack polynomials.

Let us finally provide a formula for $e(T_{\xi_\lambda}L^\lambda\Sigma)$ (see Appendix for the proof):

Proposition 4.4. *The equivariant Euler class of the tangent space to $L^\lambda\Sigma$ at ξ_λ equals*

$$e(T_{\xi_\lambda}L^\lambda\Sigma)=\prod_{\Box\in\lambda}((l(\Box)+1)\epsilon_1-a(\Box)\epsilon_2)=\epsilon_1^{|\lambda|}\cdot c_\lambda(k).$$

Remark 4.6. Note that $\epsilon_1^{-|\lambda|}\cdot[\xi_\lambda]$ corresponds to the integral form of the Jack polynomial $J_\lambda^{(k)}$.

[4] Since $H_*^{T,BM}(X^{[n]})_{\mathrm{loc}}\simeq H_*^{T,BM}((X^{[n]})^T)_{\mathrm{loc}}\simeq\bigoplus_{\lambda\vdash n}\mathbb{F}\cdot[\xi_\lambda]\simeq H_*^T((X^{[n]})^T)_{\mathrm{loc}}\simeq H_*^T(X^{[n]})_{\mathrm{loc}}$.

[5] For the first one we use the projection formula: $\langle Z_\alpha[i]u,v\rangle=\Pi_*(p_1^*(v)\cap p_2^*(u)\cap\pi^*(\alpha))=\langle u,Z_\alpha[-i]v\rangle$, where p_1,p_2,p_3,Π are the projections of $Z^n[i]$ to $X^{[n]},X^{[n+i]},X,\mathrm{pt}$, respectively.

[6] Let Σ' be the y-axis. By the fixed point formula: $[\Sigma]=\frac{[\mathrm{pt}]}{\epsilon_1},[\Sigma']=\frac{[\mathrm{pt}]}{\epsilon_2},[\Sigma]\cap[\Sigma']=[\mathrm{pt}]\Rightarrow[\Sigma]\cap[\Sigma]=\frac{\epsilon_2}{\epsilon_1}[\mathrm{pt}]$.

4.3. **Proof of Proposition 4.2.**

The main goal of this section is to provide a geometric interpretation of the dominance order on diagrams. For an ideal $I \in L^n\Sigma$, consider a sequence of vector spaces

$$V_i := (y^i)/(I \cap (y^i)), i \geq 0.$$

Note that $\dim V_0 = n, \dim V_n = 0$. Moreover, we have short exact sequences:

$$0 \to V_i \to V_{i-1} \to U_i \to 0,\ U_i := (y^{i-1})/((y^i) + I \cap (y^{i-1})).$$

Define $\nu_i := \dim U_i$. Then $\sum \nu_i = n - 0 = n$ and it is clear that $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n \geq 0$.[7]

Let $V^\nu \subset L^n\Sigma$ be the locus of those ideals such that the associated partition equals ν. This yields one more decomposition of $L^n\Sigma$:

$$L^n\Sigma = \bigsqcup_{\nu \vdash n} V^\nu.$$

Note that $\dim V_i \leq l$ is a closed condition for any integer l. Combining this with the formula $\dim V_i = \nu_{i+1} + \nu_{i+2} + \ldots = n - (\nu_1 + \ldots + \nu_i)$, we get

$$\overline{V^\nu} \subset \bigcup_{\nu' \geq \nu} V^{\nu'}. \tag{3}$$

Let us now establish the connection between $\{V^\mu\}_{\mu \vdash n}$-stratification of $L^n\Sigma$ and $\{L^\lambda\Sigma\}_{\lambda \vdash n}$.

Proposition 4.5. [N2, Proposition 4.14] *We have* $L^\lambda\Sigma = \overline{V^{\lambda^*}}$.

Note that the partition ν associated to ξ_μ equals $\nu = \mu^*$. We also have $\mu \leq \lambda \iff \mu^* \geq \lambda^*$.[8] These observations together with Proposition 4.5 and (3) imply Proposition 4.2.

Proof of Proposition 4.5.

According to Proposition 3.6, we can view $L^\lambda\Sigma$ as a closure of W^-_λ. For a generic point $\xi = [I] \in W^-_\lambda$, we have $\lim_{t\to\infty} (1,t) \cdot I = I_{\lambda_1,z_1} \cap \cdots \cap I_{\lambda_r,z_r}$, where $z_1, \ldots, z_r$ are pairwise distinct points of Σ and $\lambda_1, \ldots, \lambda_n$ are 1-column Young diagrams. It is clear that the partition $\nu = \nu(\lambda_j)$ corresponding to I_{λ_j,z_j} is just $\nu(\lambda_j) = (1^{\lambda_j})$, i.e., $I_{\lambda_j,z_j} \in V^{(1^{\lambda_j})}$.

Since the support $\mathrm{supp}((1,t)\cdot\xi) \subset \Sigma$ is independent of t, we get

$$I = I_1 \cap \cdots \cap I_r \text{ with } \mathrm{supp}(\mathbb{C}[x,y]/I_j) = \{(z_j, 0)\}.$$

On the other hand, $V^{(1^{\lambda_j})}$ is an open stratum of $L^{\lambda_j}\Sigma$, due to (3). Therefore $(1,t)\cdot I_j \in V^{(1^{\lambda_j})}$ for "sufficiently large" t. Notice also that $V^{(1^{\lambda_j})}$ is T'-invariant. Therefore

$$I_j \in V^{(1^{\lambda_j})} \implies I \in V^{\lambda^*} \implies L^\lambda\Sigma \subseteq \overline{V^{\lambda^*}}.$$

Conversely, given a point $\xi = [I] \in V^{\lambda^*}$ we have $\lim_{t\to\infty} (1,t)\cdot I = \bigoplus (I \cap (y^{i-1}))/(I \cap (y^i)) =: I_\infty$. Obviously $I_\infty \in S^\lambda\Sigma \implies I \in W^-_\lambda \implies \overline{V^{\lambda^*}} \subseteq L^\lambda\Sigma$.

The result follows. □

Remark 4.7. During the proof, we saw that $V^{(1^n)}$ is an open stratum of $L^n\Sigma$. Let us point out that $L^{(1^n)}\Sigma$ also has a simple description: $L^{(1^n)}\Sigma \simeq \Sigma^{[n]} \simeq \mathrm{Sym}^n\,\Sigma$.

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[7] If the images of $\{f_k(x)y^{i-1}\}_{k=1}^l$ are linearly independent in U_i, then the images of $\{f_k(x)y^{i-2}\}_{k=1}^l$ are also linearly independent in U_{i-1}.

[8] To prove this assume the contrary: there exist λ, μ such that $\mu \leq \lambda$, but $\mu^* \not\geq \lambda^*$. The latter condition implies an existence of r such that $\mu_1^* + \ldots + \mu_j^* \geq \lambda_1^* + \ldots + \lambda_j^*$ for $j < r$, but $\mu_1^* + \ldots + \mu_r^* < \lambda_1^* + \ldots + \lambda_r^*$. In particular, $\mu_r^* < \lambda_r^*$ and $\mu_{r+1}^* + \mu_{r+2}^* + \ldots > \lambda_{r+1}^* + \lambda_{r+2}^* + \ldots$. The latter inequality can be rewritten as $(\mu_1 - r) + \ldots + (\mu_{\mu_r^*} - r) > (\lambda_1 - r) + \ldots + (\lambda_{\lambda_r^*} - r)$, which contradicts $\mu \leq \lambda$.

Appendix A. Character formula and the Euler classes

In this appendix we prove the character formula (†) by realizing the tangent space $T_{\xi_\lambda}(\mathbb{C}^2)^{[n]}$ as the middle homology of an explicit complex of T-representations. As a corollary of this formula, we deduce Proposition 4.4 as well as the *norm formula* for the Jack polynomials.

A.1. **The character formula.**

Let $V_n := \mathbb{C}^n$ and identify $\mathfrak{gl}_n$ with $\mathrm{End}(V_n)$. Recall that $(\mathbb{C}^2)^{[n]} = \widetilde{\mathfrak{M}}_n/\operatorname{GL}_n$, where

$$\widetilde{\mathfrak{M}}_n = \{(A,B,i,j) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathrm{Hom}(\mathbb{C},V_n) \times \mathrm{Hom}(V_n,\mathbb{C}) \mid [A,B]+ij=0,\ \mathbb{C}[A,B](\operatorname{Im} i)=V_n\}.$$

The action of $G=\operatorname{GL}_n$ on $\widetilde{\mathfrak{M}}_n$ is given by $g(A,B,i,j)=(gAg^{-1},gBg^{-1},gi,jg^{-1}),\ g\in G$.

We view $\mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathrm{Hom}(\mathbb{C},V_n) \times \mathrm{Hom}(V_n,\mathbb{C})$ as the cotangent bundle of $\mathfrak{gl}_n \times \mathrm{Hom}(V_n,\mathbb{C})$, while the map $\mu:(A,B,i,j)\mapsto [A,B]+ij\in\mathfrak{gl}_n$ is the moment map for the above G-action. We also identify $T_{\mathrm{Id}}G\simeq\mathfrak{gl}_n,\ T_{\xi_0}\widetilde{\mathfrak{M}}_n\simeq\mathfrak{gl}_n\times\mathfrak{gl}_n\times V_n\times V_n^*$ for any point $\xi_0=(A_0,B_0,i_0,j_0)\in\widetilde{\mathfrak{M}}_n$.

The differential of the G-action in the neighborhood of $\xi_0\in\widetilde{\mathfrak{M}}_0$ is given by[9]

$$dm^{\xi_0}:\mathfrak{gl}_n\to\mathfrak{gl}_n\times\mathfrak{gl}_n\times V_n\times V_n^*,\ Z\mapsto([Z,A_0],[Z,B_0],Zi_0,-j_0Z=0).$$

This map is injective. Indeed, if Z is mapped to zero, then $i_0\in\operatorname{Ker}(Z)$ and so $\operatorname{Ker}(Z)\neq 0$. But $\operatorname{Ker}(Z)$ is stable with respect to A,B and hence must be the whole space V_n, i.e., $Z=0$.

The differential $d\mu_{\xi_0}:\mathfrak{gl}_n\times\mathfrak{gl}_n\times V_n\times V_n^*\to\mathfrak{gl}_n$ of the moment map is given by

$$d\mu_{\xi_0}:(A,B,i,j)\mapsto[A_0,B]+[A,B_0]+i_0j.$$

Identifying $\operatorname{Coker}(d\mu_{\xi_0})\simeq\operatorname{Im}(d\mu_{\xi_0})^\perp$ with respect to the trace form, we get:

$$\operatorname{Coker}(d\mu_{\xi_0})=\{C\in\mathfrak{gl}_n\mid \operatorname{tr}(C[A_0,B]+C[A,B_0]+Ci_0j)=0\ \ \forall A\in\mathfrak{gl}_n,B\in\mathfrak{gl}_n,j\in V_n^*\}=$$
$$\{C\in\mathfrak{gl}_n\mid [C,A_0]=[C,B_0]=0,Ci_0=0\}=0,$$

where we used the stability condition in the last equality. Thus, $d\mu_{\xi_0}$ is actually surjective.

Hence, we get a complex

$$(\ddagger)\ \ \mathrm{Hom}(V_n,V_n)\overset{a}{\hookrightarrow}\mathrm{End}(V_n,V_n)\oplus\mathrm{End}(V_n,V_n)\oplus\mathrm{Hom}(V_n,\mathbb{C})\oplus\mathrm{Hom}(\mathbb{C},V_n)\overset{b}{\twoheadrightarrow}\mathrm{Hom}(V_n,V_n),$$

where $a:=dm^{\xi_0},\ b:=d\mu_{\xi_0}$. The middle homology of it equals

$$\operatorname{Ker}(b)/\operatorname{Im}(a)\simeq T_{\bar\xi_0}(\mathbb{C}^2)^{[n]},\ \text{where } \bar\xi_0\in X^{[n]} \text{ is the image of } \xi_0\in\widetilde{\mathfrak{M}}_n.$$

To compute the T-character of $T_{\xi_\lambda}(\mathbb{C}^2)^{[n]}$, we should view (‡) as a complex of T-representations. Recall that $V_n\simeq Q_\lambda:=\mathbb{C}[x,y]/I_\lambda$, where the operators A,B correspond to the multiplications by x,y. Hence, the natural T-weight decomposition of Q_λ corresponds to the T-weight decomposition $V_n=\bigoplus_{k,l}V_n(k,l)$ with $\operatorname{Im}(i)\in V_n(0,0)$ and $\deg(A)=(-1,0),\deg(B)=(0,-1)$.

Let us rewrite the above complex by changing the middle term to

$$C_2:=\mathrm{Hom}(V_n,V_n\otimes Q)\oplus\mathrm{Hom}(\mathbb{C},V_n)\oplus\mathrm{Hom}(V_n,\mathbb{C}\otimes\wedge^2Q),$$

the rightmost term to $C_1:=\mathrm{Hom}(V_n,V_n)\otimes\wedge^2Q$, the leftmost term to $C_3:=\mathrm{Hom}(V_n,V_n)$, where Q is the 2-dimensional T-module and the maps $C_3\to C_2\to C_1$ are the same.

This yields the complex of T-representations

$$0\to C_3\to C_2\to C_1\to 0.$$

Identifying the tangent space $T_{\xi_\lambda}(\mathbb{C}^2)^{[n]}$ with the middle homology of this complex, we get

$$\operatorname{ch}T_{\xi_\lambda}(\mathbb{C}^2)^{[n]}=\operatorname{ch}(C_2)-\operatorname{ch}(C_1)-\operatorname{ch}(C_3)=\operatorname{ch}\left(V_n^*\otimes V_n\otimes(Q-\wedge^2Q-1)+V_n+V_n^*\otimes\wedge^2Q\right).$$

Exercise A.1. *Derive (†) by using* $\operatorname{ch}(Q)=t_1+t_2,\ \operatorname{ch}(V_n)=\sum_{i=1}^{l(\lambda)}\sum_{j=1}^{\lambda_i}t_1^{1-i}t_2^{1-j}$.

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[9] Recall that the stability condition forces $j_0=0$.

A.2. **Proof of Proposition 4.4.**

It is easy to see that $L^\lambda\Sigma$ is a submanifold in a neighborhood of $\{\xi_\lambda\}$. Due to Proposition 3.8, the tangent space $T_{\xi_\lambda}(L^\lambda\Sigma)$ is the direct sum of negative $T_{m,l}$-weight subspaces of $T_{\xi_\lambda}(X^{[n]})$ for generic m, l. Combining this observation with (†), we get

Corollary A.2. *We have* ch $T_{\xi_\lambda}(L^\lambda\Sigma) = \sum_{\square\in\lambda} t_1^{l(\square)+1} t_2^{-a(\square)}$.

This corollary implies Proposition 4.4.

We conclude this appendix with the following result:

Proposition A.3. *The norm of the Jack polynomial is given by*

$$\langle P_\lambda^{(k)}, P_\lambda^{(k)}\rangle_k = \prod_{\square\in\lambda} \frac{l(\square) + k\cdot(a(\square)+1)}{l(\square)+1+k\cdot a(\square)}.$$

Proof. According to Theorem 4.3, the isomorphism θ^T intertwines pairing $\langle,\rangle_k$ with $\langle,\rangle$ and

$$\theta^T : P_\lambda^{(k)} \to e(T_{\xi_\lambda} L^\lambda\Sigma)^{-1}[\xi_\lambda].$$

Therefore, we get

$$\langle P_\lambda^{(k)}, P_\lambda^{(k)}\rangle_k = \frac{1}{e(T_{\xi_\lambda} L^\lambda\Sigma)^2} \langle [\xi_\lambda], [\xi_\lambda]\rangle = (-1)^{|\lambda|} \frac{e(T_{\xi_\lambda} X^{[|\lambda|]})}{e(T_{\xi_\lambda} L^\lambda\Sigma)^2}.$$

It remains to use the equality $k = -\epsilon_2/\epsilon_1$, Proposition 4.4 and the formula

$$e(T_{\xi_\lambda} X^{[n]}) = \prod_{\square\in\lambda} ((l(\square)+1)\epsilon_1 - a(\square)\epsilon_2)(-l(\square)\epsilon_1 + (a(\square)+1)\epsilon_2).$$

$\square$

References

[LQW] W. Li, Z. Qin, and W. Wang, The cohomology rings of Hilbert schemes via Jack polynomials, CRM Proc. Lecture Notes, Vol. 38, Amer. Math. Soc., Providence (2004), 249–258; arXiv/0411255.

[M] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Math. Monographs, Oxford Univ. Press (1995), ISBN 0-19-853489-2.

[N1] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, Univ. Lecture Ser. **18**, Amer. Math. Soc., Providence (1999), ISBN 0-8218-1956-9.

[N2] H. Nakajima, *Jack polynomials and Hilbert schemes of points on surfaces*, arXiv/9610021.

[N3] H. Nakajima, *More lectures on Hilbert schemes of points on surfaces*, arXiv/1401.6782.

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